ABSOLUTE CONTINUITY ON PATHS OF SPATIAL OPEN DISCRETE MAPPINGS

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Abstract. We prove that open discrete mappings of Sobolev classes $W_{\rm loc}^{1,p}$, $p > n-1$, with locally integrable inner dilatations admit ACP_p^{-1}-property, which means that these mappings are absolutely continuous on almost all preimage paths with respect to p-module. In particular, our results extend the well-known Poletskiĭ lemma for quasiregular mappings. We also establish the upper bounds for p-module of such mappings in terms of integrals depending on the inner dilatations and arbitrary admissible functions.

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1. Introduction

1.1. The absolute continuity on almost all lines of mappings is one of crucial properties of quasiconformal mappings and more general quasiregular mappings (see, e.g. [16], [21], [22], [28], [29]). Later many authors have extended this important property to more general classes of continuous mappings of finitely dimensional domains including mappings of finite distortion, mappings with finite length distortion, mappings quasiconformal in the mean, mappings with controlled p-module and others ([1], [3], [8], [10], [12], [13], [14], [18], [19], [24]). It remains still open whether the generic mappings from Sobolev classes possess absolute continuity and related differential properties (cf. [2], [9], [15], [20]).

In this paper, we discuss absolute continuity of Sobolev mappings on paths. Due to terminology of [18], this class $ACP \subset ACL$, where ACL denotes the class of mappings absolutely continuous on lines. The ACP-property is a basic tool for studying geometric features of mappings (cf. [1], [7]). It was shown in [20], that quasiregular mappings provide absolute continuity for preimages; namely, if $f : D \to \mathbb{R}^n$, $n \geq 2$, is a quasiregular mapping of a domain $D \subset \mathbb{R}^n$, then for almost all curves $\gamma_* \in f(D)$ a curve γ, such that $f \circ \gamma = \gamma_*$, is absolutely continuous. This property can be treated as ACP^{-1}-property. For this property and the definition of mappings with finite length distortion we refer to [17] and [18, Ch. 8].

An extension of Poletskiĭ's result to general open discrete mappings of Sobolev class $W_{\rm loc}^{1,n}(D)$ is given [25], under assumption that the n-dimensional Lebesgue measure of the branching set $m(B_f) = 0$ and local integrability with appropriate degree of one of quasiconformality coefficients of mappings. Other resent extensions related to absolute continuity can be found in [4], [5], [11], [23], [24] and [26].

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1.2. Throughout this paper, D will be a domain in $\mathbb{R}^n$, $n \geq 2$; m denotes the Lebesgue measure in $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^n$ is *discrete* if $f^{-1}(y)$ consists of isolated points for each $y \in \mathbb{R}^n$, and f is *open* if it maps open sets onto open sets. The notation $f : D \to \mathbb{R}^n$ assumes that f is continuous, and all mappings are orientation preserving, i.e., the topological index $i(y, f, G) > 0$ for an arbitrary domain $G \Subset D$ and $y \in f(G) \setminus f(\partial G)$; see, e.g. [21, II.2]. Let $f : D \to \mathbb{R}^n$ be a mapping and suppose that there is a domain $G \subset D$, $\overline{G} \subset D$, for which $f^{-1}(f(x)) = \{x\}$. Then the quantity $\mu(f(x), f, G)$, which is referred to be the local topological index, which does not depend on the choice of G and is denoted by $i(x, f)$.

A curve γ in $\mathbb{R}^n$ ($\overline{\mathbb{R}^n}$) is a continuous mapping $\gamma : I \to \mathbb{R}^n$ ($\overline{\mathbb{R}^n}$), where I is an interval in $\mathbb{R}$. Its locus $\gamma(I)$ is denoted by $|\gamma|$. Let Γ be a family of curves γ in $\mathbb{R}^n$. A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ (abbr. $\rho \in \operatorname{adm} \Gamma$) if $\int\limits_{\gamma} \rho(x)|dx| \geq 1$ for each locally rectifiable $\gamma \in \Gamma$. For $p \geq 1$, we define the quantity

$$\mathcal{M}_p(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x) dm(x),$$

which is called *p-module* of Γ [28, 6.1] (see also [22], [29]).

For the points of differentiability of f we define

$$l(f'(x)) = \min_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}, ||f'(x)|| = \max_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}, J(x, f) = \det f'(x),$$

and for any $x \in D$ and fixed $p \geq 1$ the p-inner dilatation of f by

$$K_{I,p}(x, f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^p}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise}. \end{cases}$$

The notation $K_I(x, f) = K_{I,n}(x, f)$ relates to the classical inner dilatation coefficient.

Recall that a mapping $f : D \to \mathbb{R}^n$ is said to possess Lusin's *N-property* if $m(f(S)) = 0$ whenever $m(S) = 0$ for $S \subset \mathbb{R}^n$. Similarly, f has *N^{-1}–property* if $m(f^{-1}(S)) = 0$ whenever $m(S) = 0$.

1.3. Let Γ be a curve family in D, Γ' be a curve family in $\mathbb{R}^n$ and q be a positive integer such that the following is true. Suppose that for every curve $\beta : I \to D$ in Γ' there exist curves $\alpha_1, \ldots, \alpha_q$ in Γ such that $f \circ \alpha_j \subset \beta$ for all $j = 1, \ldots, q$, and for every $x \in D$ and all $t \in I$ the equality $\alpha_j(t) = x$ holds at most $i(x, f)$ indices j.

Theorem 1.1. *Let $f : D \to \mathbb{R}^n$ be an open discrete mapping of the class $W^{1,p}_{\mathrm{loc}}$, $n - 1 < p \leq n$, for which N and N^{-1}-properties hold. Suppose that $K_I(x, f) \in L^1_{\mathrm{loc}}$. Then*

$$\mathcal{M}_p(\Gamma') \leq \frac{1}{q} \int\limits_D K_{I,p}(x, f) \cdot \rho^p(x)\, dm(x) \tag{1}$$

for every $\rho \in \operatorname{adm} \Gamma$. In particular,

$$\mathcal{M}_p(f(\Gamma)) \leq \int\limits_D K_{I,p}(x, f) \cdot \rho^p(x)\, dm(x)$$

for every family of curves Γ in D and $\rho \in \operatorname{adm} \Gamma$.

Theorem 1.2. *Let $f : D \to \mathbb{R}^n$ be an open discrete mapping of the class $W^{1,p}_{\rm loc}$, $p > n$, for which N^{-1}-property holds. Assume that $K_I(x, f) \in L^1_{\rm loc}$ and $K_{I,p}(x, f) \in L^1_{\rm loc}$. Then the inequality (1) holds.*

2. Auxiliary results

Let us first recall some necessary definitions, notations and statements.

2.1. We say that a property $\mathcal{P}$ holds for *p-almost every (p-a.e.)* curves γ if $\mathcal{P}$ holds for all curves except a family of zero p-module.

If $\gamma : I \to \mathbb{R}^n$ is a locally rectifiable curve, then there is a unique nondecreasing length function l_γ of I onto a length interval $I_\gamma \subset \mathbb{R}$ with a prescribed normalization $l_\gamma(t_0) = 0 \in I_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the arc $\gamma|_{[t_0,t]}$ of γ for $t > t_0$, $t \in I$, and $l_\gamma(t)$ is equal to minus length of $\gamma|_{[t,t_0]}$ when $t < t_0$, $t \in I$. Let $g : |\gamma| \to \mathbb{R}^n$ be a continuous mapping and $\widetilde{\gamma} = g \circ \gamma$ be locally rectifiable. Then there is a unique nondecreasing function $L_{\gamma,g} : I_\gamma \to I_{\widetilde{\gamma}}$ such that $L_{\gamma,g}\left(l_\gamma(t)\right) = l_{\widetilde{\gamma}}(t)$ for all $t \in I$. The curve γ in D is called a (whole) *lifting* of the curve $\widetilde{\gamma}$ in $\mathbb{R}^n$ under $f : D \to \mathbb{R}^n$ and $\widetilde{\gamma} = f \circ \gamma$.

Following [24], we say that a mapping $f : D \to \mathbb{R}^n$ satisfies the $L^{(2)}_p$*-property,* if, for p-a.e. curve $\widetilde{\gamma}$ in $f(D)$, each lifting γ of $\widetilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,f}$ satisfies N^{-1}-property.

For any closed rectifiable path $\gamma : [a, b] \to \mathbb{R}^n$, we define the length function $l_\gamma(t)$ by $l_\gamma(t) = S\left(\gamma, [a, t]\right)$, where $S(\gamma, [a, t])$ is the length of the path $\gamma|_{[a,t]}$. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a rectifiable curve in $\mathbb{R}^n$, $n \geq 2$, and $l(\alpha)$ be its length. A *normal representation* α^0 of α is defined as a curve $\alpha^0 : [0, l(\alpha)] \to \mathbb{R}^n$ which is obtained from α by changing parameter so that $\alpha(t) = \alpha^0\left(S\left(\alpha, [a, t]\right)\right)$ for every $t \in [0, l(\alpha)]$.

Suppose that α and β are curves in $\mathbb{R}^n$. Then the notation $\alpha \subset \beta$ means that α is a subpath of β. In what follows, I denotes either an open or closed or semi-open interval on the real axes. The following definition is due to [22, Ch. II].

Let $f : D \to \mathbb{R}^n$ be a mapping whose inverse $f^{-1}(y)$ does not contain a nondegenerate curve, $\beta : I_0 \to \mathbb{R}^n$ be a closed rectifiable curve and $\alpha : I \to D$ be such that $f \circ \alpha \subset \beta$. If the length function $l_\beta : I_0 \to [0, l(\beta)]$ is constant on $J \subset I$, then β is also constant on J and, consequently, the curve α is constant on J too. Thus, there is a unique function $\alpha^* : l_\beta(I) \to D$ such that $\alpha = \alpha^* \circ (l_\beta|_I)$. We say that α^* is the *f–representation of α with respect to β* if $\beta = f \circ \alpha$.

Remark 2.1. Note that properties of $L_{\gamma,f}$ connected with the length functions $l_\gamma(t)$ and $l_{\widetilde{\gamma}}(t)$, $\widetilde{\gamma} = f \circ \gamma$, do not essentially depend on the choice of $t_0 \in [a, b)$. Moreover, one can take $t_0 = a$, because for any $t_0 \in (a, b)$, we have $S(\gamma, [a, t]) = S(\gamma, [a, t_0]) + l_\gamma(t)$. Further we put $t_0 = a$ and use the notion $l_\gamma(t)$ for the length of the path $\gamma|_{[a,t]}$ whenever the curve γ is closed.

Given a set E in $\mathbb{R}^n$ and a curve $\gamma : I \to \mathbb{R}^n$, we identify $\gamma \cap E$ with $\gamma\,(I) \cap E$. If γ is locally rectifiable, we set

$$l\left(\gamma \cap E\right) = m_1(E_\gamma),$$

where $E_\gamma = l_\gamma\left(\gamma^{-1}(E)\right)$. Note that $E_\gamma = \gamma_0^{-1}(E)$, where $\gamma_0 : I_\gamma \to \mathbb{R}^n$ is the natural parametrization of γ and

$$l(\gamma \cap E) = \int\limits_I \chi_E(\gamma(t))\,|dx| := \int\limits_{I_\gamma} \chi_{E_\gamma}(s)\,ds\,.$$

2.2. We shall use the following result established in [24].

Proposition 2.1. *A mapping $f : D \to \mathbb{R}^n$ has the $L_p^{(2)}$–property if and only if $f^{-1}(y)$ does not contain a nondegenerate curve for every $y \in \mathbb{R}^n$, and the f-representation γ^* is rectifiable and absolutely continuous for p-a.e. closed curve $\widetilde{\gamma} = f \circ \gamma$.*

The following three statements can be found in [20]; see Lemmas 1–3.

Proposition 2.2. *Let $\gamma_1 : I = [0, l] \to \mathbb{R}^n$ be a rectifiable curve and let $B = \overline{B} \subset I$, $l_{\gamma_1}(B) = 0$. Suppose that $\gamma_2 : I \to \mathbb{R}^n$ is a rectifiable curve on $I \setminus B$, and $\gamma_1(t) = \gamma_2(t)$ for $t \in B$. Then γ_2 is also rectifiable and $l_{\gamma_2}(B) = 0$.*

Proposition 2.3. *Let $\gamma : I = [0, l] \to \mathbb{R}^n$ be a curve, $B = \overline{B} \subset I$, and let $E \subset I$ be a set with $\overline{E} \subset E \cup B$, and $E \cap B = \varnothing$. If γ is rectifiable on $I \setminus (E \cup B)$, and moreover, for every point $t \in I \setminus B$ there exists a neighborhood V, in which γ is rectifiable and $l_\gamma(V) = l_\gamma(V \setminus E)$, then γ is rectifiable on $I \setminus B$ and $l_\gamma(I \setminus B) = l_\gamma\left(I \setminus (E \cup B)\right)$.*

Proposition 2.4. *Let $\gamma : I \to \mathbb{R}^n$ be a rectifiable curve. If $l_\gamma(B) = 0$ for every $B \subset I$ with $\mathrm{m}_1(B) = 0$, then $l_\gamma(t)$ is absolutely continuous function.*

2.3. Define for a mapping $f : D \to \mathbb{R}^n$, a set $E \subset D$ and a point $y \in \mathbb{R}^n$, the multiplicity function $N(y, f, E)$ as the number of preimages of y in E, i.e.,

$$N(y, f, E) = \text{card}\,\{x \in E : f(x) = y\}\,,$$

and put

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E)\,.$$

Recall that a domain $G \Subset D$ is said to be *normal domain of* f, if $\partial f(G) = f(\partial G)$. If G is a normal domain, then $\mu(y, f, G)$ is a constant for any $y \in f(G)$. This constant will be denoted by $\mu(f, G)$. Let $f : D \to \mathbb{R}^n$ be a discrete open mapping, then $\mu(f, G) = N(f, G)$ for every normal domain $G \subset D$, see, e.g. [22, Proposition I.4.10].

Let $V \subset D$ be a normal domain, and $f(V) = V^*$. Following [20], we define $g_V : V^* \to \mathbb{R}^n$ as follows: if $y \in V^*$, and $f^{-1}(y) \cap V = \{x_i\}$, then

$$g_V(y) = \frac{1}{q}\sum_i i(x_i, f)x_i\,, \tag{2}$$

where $q = \sum i(x_i, f) = \mu(f, V)$. This mapping can be regarded as the inverse for the nonhomeomorphic mapping f. The following statement is due to [11].

Proposition 2.5. *Let $f : D \to \mathbb{R}^n$ be an open discrete mapping, $f \in W_{loc}^{1,n-1}(D)$, obeying $K_I(x, f) \in L_{loc}^1$, and $m(f(B_f)) = 0$. Then $g_V(y)$ is continuous V^* and $g_V(y) \in ACL^n(V^*)$.*

Similar to p-inner dilatation, we define for any $x \in D$ and fixed $p \geq 1$ the p-outer dilatation of f by

$$K_{O,p}(x,f) \quad = \quad \begin{cases} \frac{\|f'(x)\|^p}{|J(x,f)|}, & J(x,f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

In the case $p = n$ the classical outer dilation coefficient coincides with p-outer dilatation, i.e. $K_O(x,f) = K_{O,n}(x,f)$.

3. Main results

Poletskiĭ lemma's yields that ACP^{-1}-property can be regarded as the well-known Fuglede theorem in the "inverse direction". Fuglede's theorem states that for a continuous mapping $f : D \to \mathbb{R}^n$ of Sobolev class $W^{1,p}_{\text{loc}}(D)$ and a family Γ of all locally rectifiable curves in D having a closed subpath on which f is not absolutely continuous, Γ is exceptional with respect to p-module, i.e. $\mathcal{M}_p(\Gamma) = 0$. The following theorem extends Poletskiĭ lemma's to the mappings satisfying the assumptions of Theorems 1.1 or 1.2. This theorem involves the ACP_p^{-1}-property.

Theorem 3.1. *Let $f : D \to \mathbb{R}^n$ be an open discrete mapping satisfying the assumptions either of Theorem 1.1 or of Theorem 1.2. Then $f \in ACP_p^{-1}$.*

Proof. In view of Proposition 2.1, it suffices to prove that for any p-a.e. closed curve $\widetilde{\gamma}$ such that $f \circ \gamma = \widetilde{\gamma}$, the f-representation γ^* of γ with respect to $\widetilde{\gamma}$ is rectifiable and absolutely continuous. We also can restrict ourselves by considering a subfamily of $\widetilde{\Gamma}$, which belongs to a compact subdomain D' of the domain D. The general case is obtained from this by exhausting $\{V_i\}_{i=1}^{\infty}$ of the domain $f(D)$ by compact subdomains $V_i \Subset D$.

Denote by $\widetilde{\gamma}^0$ the normal representation of a closed rectifiable curve $\widetilde{\gamma}$, which lies in D', $\widetilde{\gamma} = f \circ \gamma$, and by γ^* the f-representation of γ with respect to $\widetilde{\gamma}$. Now $\widetilde{\gamma}^0 : [0, l(\gamma)] \to \mathbb{R}^n$, $\gamma^* : [0, l(\gamma)] \to \mathbb{R}^n$. Set $I := [0, l(\gamma)]$.

We show that for a.e. closed curve $\widetilde{\gamma}$, the curve γ^* is rectifiable on $I \setminus \gamma^*(B_f)$, where $\gamma^*(B_f) = \{s : \gamma^*(s) \in B_f\}$. Let the sets $D' \setminus B_f$ be covered by a countable system of neighborhoods $\{A_l\}$ at which the corresponding mapping $f_l = f|_{A_l}$ is homeomorphic. Put $h_l = f_l^{-1}$. Since under assumptions of Theorem 3.1 $f \in W^{1,p}_{\text{loc}}$, $p > n-1$, one concludes that $h_l \in W^{1,1}_{\text{loc}}$ ([30, Theorem 1.1]).

The following lemma shows that the mapping h_l has richer differential properties.

Lemma 3.1. *Let $f : D \to \mathbb{R}^n$ be an open discrete mapping satisfying the assumptions either of Theorem 1.1 or of Theorem 1.2. Then $h_l \in W^{1,p}_{\text{loc}}$.*

Proof of Lemma 3.1. By the assumptions, f has N-property, therefore, $J(y, h_l) \neq 0$ for a.e. $y \in f(A_l)$. Consequently, $\|h_l'(y)\|^n = K_O(y, h_l) \cdot |J(y, h_l)|$ for a.e. $y \in f(A_l)$.

Consider first the case $p\le n$. By [6, Theorem 3.2.5] and the features of N^{-1}-property for f, we have

$$\begin{aligned}\int\limits_{f(A_l)}\|h_l^{\prime}(y)\|^n\,dm(y)&=\int\limits_{f(A_l)}K_O(y,h_l)\,|J(y,h_l)|\,dm(y)\\&=\int\limits_{f(A_l)}K_O(h_l^{-1}(h_l(y)),h_l)\,|J(y,h_l)|\,dm(y)\\&=\int\limits_{A_l}K_O(h_l^{-1}(x),h_l)\,dm(x)\,.\end{aligned}\tag{3}$$

Since $f\in W_{\rm loc}^{1,p}$, $p>n-1$, and f is open, it is differentiable a.e.; see e.g. [27, Lemma 3]. Further, $h_l^{\prime}(h_l^{-1}(x))=g^{\prime}(f(x))$, hence, $h_l^{\prime}(f(x))=(f^{\prime}(x))^{-1}$ at all points of nondegenerate differentiability of f. Since f possesses N^{-1}-property, we have $J(x,f)\ne0$ a.e., and moreover,

$$\|h_l^{\prime}\,(f(x))\,\|=\frac{1}{l\,(f^{\,\prime}(x))}\,,\qquad J(f(x),h_l)=\frac{1}{J(x,f)}\,;\tag{4}$$

see [21, Section 4.I]. Applying (3) and (4), one gets

$$\int\limits_{f(A_l)}\|h_l^{\prime}(y)\|^n\,dm(y)=\int\limits_{A_l}K_I(x,f)\,dm(x)\,.$$

Thus, $h_l\in W_{\rm loc}^{1,n}$, because $h_l\in W_{\rm loc}^{1,1}$. In particular, we proved that $h_l\in W_{\rm loc}^{1,p}$, for $n-1<p\le n$.

Now let $p>n$. Since f satisfies N-property, $J(y,h_l)\ne0$ for a.e. $y\in f(A_l)$. Consequently, $\|h_l^{\prime}(y)\|^p=K_{O,p}(y,h_l)\cdot|J(y,h_l)|$ for a.e. $y\in f(A_l)$. Arguing similarly to above, one obtains

$$\begin{aligned}\int\limits_{f(A_l)}\|h_l^{\prime}(y)\|^p\,dm(y)&=\int\limits_{f(A_l)}K_{O,p}(y,h_l)\,|J(y,h_l)|\,dm(y)\\&=\int\limits_{f(A_l)}K_{O,p}(h_l^{-1}(h_l(y)),h_l)\,|J(y,h_l)|\,dm(y)\\&=\int\limits_{A_l}K_{O,p}(h_l^{-1}(x),h_l)\,dm(x)\,.\end{aligned}$$

Applying (4), one concludes

$$\int\limits_{f(A_l)}\|h_l^{\prime}(y)\|^p\,dm(y)=\int\limits_{A_l}K_{I,p}(x,f)\,dm(x)\,,$$

which yields $h_l\in W_{loc}^{1,p}$ for $p>n$, because $h_l\in W_{loc}^{1,1}$. This completes the proof of Lemma 3.1. $\Box$

We proceed the proof of Theorem 3.1 and note that by [28, section 28.2], $h_l\in ACP_p$. Observe that, if $\gamma^*(s)\in A_l\cap A_j$, then $h_l\left(\widetilde{\gamma}^0(s)\right)=h_j\left(\widetilde{\gamma}^0(s)\right)$. Since $\widetilde{\gamma}^0$ is parameterized by s, one defines a mapping $g:\widetilde{\gamma}^0|_{I\setminus\gamma^*(B_f)}\to\mathbb{R}^n$ by $g(\widetilde{\gamma}^0(s))=h_k\left(\widetilde{\gamma}^0(s)\right)$, when $\gamma^*(s)\in A_k$.

Set $\frac{\partial g_l}{\partial y_j}(s) = \frac{\partial h_{kl}}{\partial y_j}(\widetilde{\gamma}(s))$. It follows from above that γ^* is locally absolutely continuous on every open interval of $I \setminus \gamma^*(B_f)$ for p-a.e. $\widetilde{\gamma} = f \circ \gamma^*$. Applying Theorem 1.3(6) from [28], one gets

$$l_{\gamma^*}(I \setminus \gamma^*(B_f)) = \int\limits_{I\setminus\gamma^*(B_f)} |\gamma^{*\prime}(s)|\; dm_1(s)$$
$$\leq \int\limits_{I\setminus\gamma^*(B_f)} \left(\sum_{l,j}\left(\frac{\partial g_l}{\partial y_j}(s)\right)^2\right)^{1/2} dm_1(s)$$

for p-a.e. $\widetilde{\gamma} = f \circ \gamma^*$. Since $h_l \in W^{1,p}_{\rm loc}$ and $m(D') < \infty$,

$$\int\limits_{I\setminus\gamma^*(B_f)} \left(\sum_{l,j}\left(\frac{\partial g_l}{\partial y_j}(s)\right)^2\right)^{1/2} dm_1(s) < \infty$$

for p-a.e. curves $\widetilde{\gamma} = f \circ \gamma^*$ (cf. [7, Theorem 3(e)]). Consequently, γ^* is rectifiable on $I \setminus \gamma^*(B_f)$ for p-a.e. curves $\widetilde{\gamma} \in \widetilde{\Gamma}$.

Moreover, $l_{\gamma^*}(C) = 0$ for p-a.e. curves $\widetilde{\gamma}$ and every set C, $C \subset I \setminus \gamma^*(B_f)$, such that $m_1(C) = 0$. In fact, $l_{\gamma^*}(C) = \int\limits_C |\gamma^{*\prime}(s)|\; dm_1(s) = 0$ for p-a.e. curves $\widetilde{\gamma} = f \circ \gamma^*$.

Now let B_l be a subset of the branched set B_f of f with $i(x, f) = l$, and

$$\gamma^*(B_l) = \{s \in I : \gamma^*(s) \in B_l\}.$$

We show that for p-a.e. $\widetilde{\gamma}$ the curve γ^* is rectifiable on $I \setminus \bigcup\limits_{k>l} \gamma^*(B_k)$ for all $l \in \mathbb{N}$ and $l_{\gamma^*}(C) = 0$ for every set C with $m_1(C) = 0$ and $C \subset I \setminus \bigcup\limits_{k>l} \gamma^*(B_k)$. This is obtained by induction with respect to l. The case $l = 1$ has been proved above.

Now we assume that this property holds for $l = j$ and show its validness for $l = j + 1$.

Since f has N^{-1}-property, $J(x, f) \neq 0$ a.e.. Hence, since f is differentiable a.e., we have $m(B_f) = 0$ (see [16, Lemma 2.14]).

Cover the set B_j by at most countable system of normal domains $\{U_l\}_{l=1}^{\infty}$ such that $\mu(f, U_l) = j$ with $\mu(f, U_l) = N(f, U_l)$ (see [16, Lemma 2.9]). By Proposition 2.5, the mapping $g_l = g_{U_l}$ defined by (2) is absolutely continuous for p-a.e. curves in $U_l^* = f(U_l)$. Observe also that $g_l(\widetilde{\gamma}^0(s)) = \gamma^*(s)$, whenever $\widetilde{\gamma}^0(s) \in f(B_j \cap U_l)$.

Letting $\alpha_{k,l} := g_l\left(\widetilde{\gamma}^{(k)}(s)\right)$, where $\widetilde{\gamma}^0|_{f(U_l)} = \bigcup\limits_{k=1}^{\infty} \widetilde{\gamma}^{(k)}(s)$, $l = 1, 2, \dots$, we have by absolute continuity of g_l on p-a.e. curves that $l_{\alpha_{k,l}}(\alpha_{k,l}(B_j \cap U_l)) = 0$ for p-a.e. $\widetilde{\gamma} = f \circ \gamma$. Now, by Proposition 2.2,

$$l_{\gamma^*}(\gamma^*(B_j \cap U_l)) = 0$$

for p-a.e. curves $\widetilde{\gamma}$ and γ. Summation over all neighborhoods U_l yields $l_{\gamma^*}(\gamma^*(B_j)) = 0$ for p-a.e. curves $\widetilde{\gamma} = f \circ \gamma^*$.

Take in Proposition 2.3 $B := \bigcup\limits_{k>j} \gamma^*(B_k)$ and $E := \gamma^*(B_j)$. The local topological index $i(x, f)$ is lower semicontinuous (see, e.g. [22, Lemma VI.8.13]). Now B is closed, and by the induction assumption γ^* is rectifiable on $I \setminus \bigcup\limits_{k>j} \gamma^*(B_k)$, and $l_{\gamma^*}(C) = 0$ for $C \subset I \setminus \bigcup\limits_{k>j} \gamma^*(B_k)$ and $m_1(C) = 0$. Since D' is a compact, there exists $M \in \mathbb{N}$ with $i(x, f) \leq M$. By the previous

step, $l_{\gamma^*}\left(\bigcup\limits_{j=2}^{M}\gamma^*(B_j)\right)=0$ for p-a.e. $\widetilde{\gamma}=f\circ\gamma^*$. Hence, by Proposition 2.4, γ^* is absolutely continuous and rectifiable for p-a.e. closed curves $\widetilde{\gamma}=f\circ\gamma^*$, $\widetilde{\gamma}\in\widetilde{\Gamma}$. This completes the proof of Theorem 3.1. $\Box$

Now both Theorems 1.1 and 1.2 follow from Theorem 3.1 and Theorem 3.1 from [24].

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